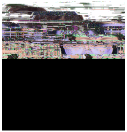
On the Maximum Size of Block Codes Subject to a Distance Criterion

Vincent Y. F. Tan
National University of Singapore (NUS)



Error-correcting codes

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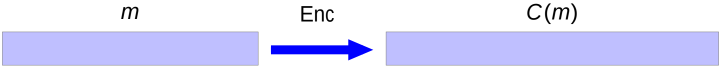

m

Enc

$C(m)$

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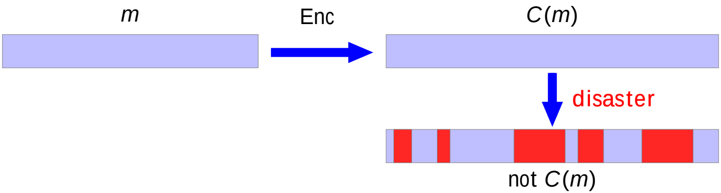

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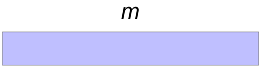

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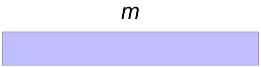

"Message" m (k symbols) maps to "codeword" $C(m)$ ($n > k$ symbols).

Set of codewords is a code C.

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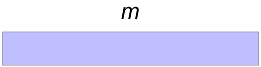

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Distance and errors

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Can correct as many errors as half the distance:

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Different “distances” for different applications.

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$$(\mathbf{x};\mathbf{y}) = \begin{cases} 0 & \mathbf{x} = \mathbf{y} \\ 1 & \text{else} \end{cases} \qquad \text{(Probability-of-error distortion)}$$

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(deletion distance, rank-metric, etc)

Coding and the distance problem

The GV bound and good codes

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Theorem (Gilbert-Varshamov bound)

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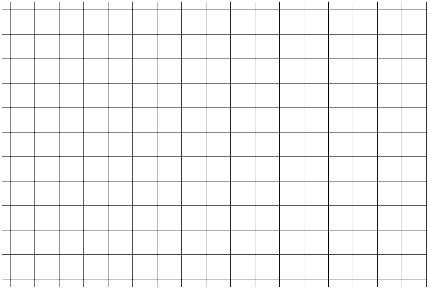

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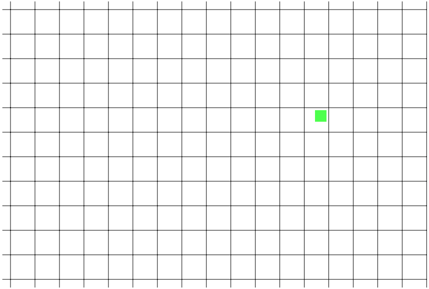

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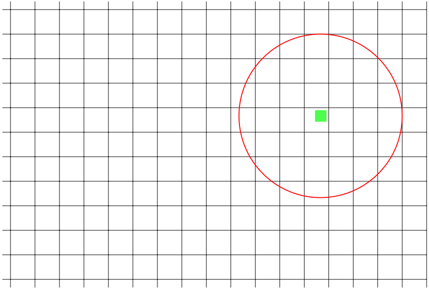

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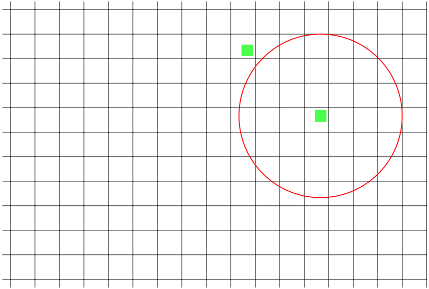

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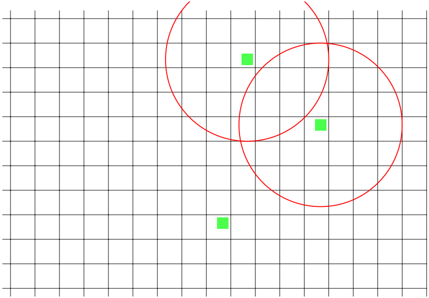

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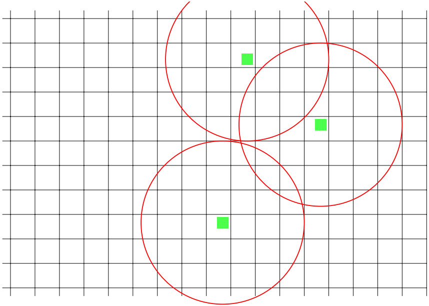

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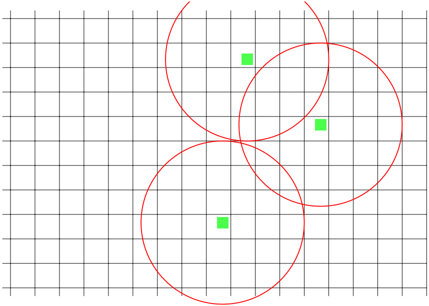

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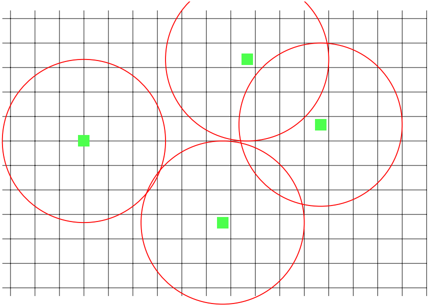

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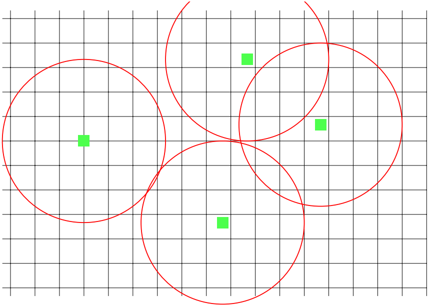

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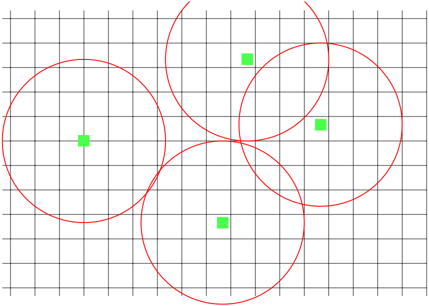

Each circle has $\ \ 2^{H(\ )n}$ vectors, so final code size is $2^n{=}2^{H(\ )n}$.

GV continued

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Pick i.i.d. codewords uniformly from $\{0,1\}^n$.

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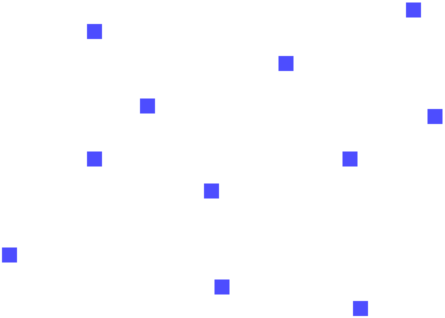

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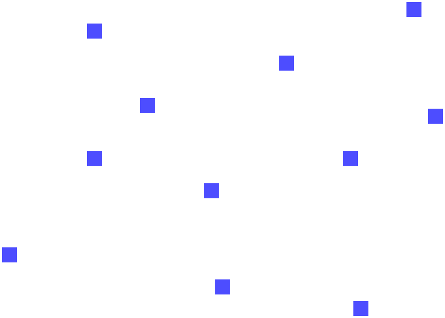

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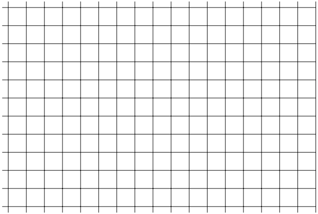

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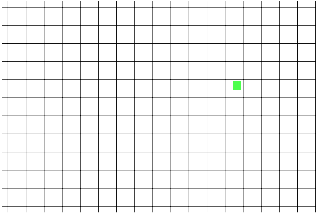

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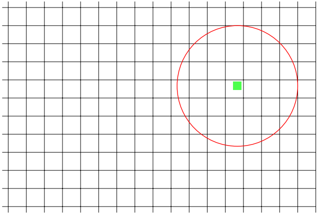

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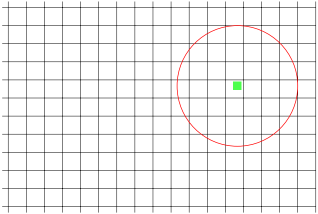

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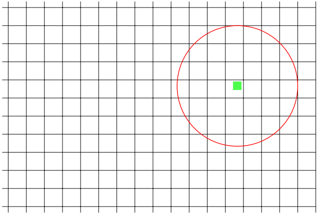

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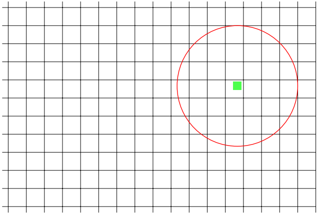

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Number of “bad” pairs $(\mathbf{x}; \mathbf{y})$ is

$$2^{2Rn} \quad \frac{2^{H(\ )n}}{2^n} = 2^{(R \quad)n}:$$

Remove one element from each bad pair.

Distance is now , and rate is still $\quad R$.

Extending GV

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Extending GV

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This work: What if we don't use the *uniform* distribution in the random proof?

(Could imagine: supported on structured set, mixing distributions.)

To mimic the GV proof, need to understand collision probability.

When are two random codewords at distance $< d$?

In other words. . .

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Moral: For various $\mathbf{X}$, want to understand collision probability (distance spectrum):

$$F_{\mathbf{X}}(d) := \Pr \quad (\mathbf{X}; \hat{\mathbf{X}}) < d \; ;$$

where $\hat{\mathbf{X}}$ is an independent copy of $\mathbf{X}$.

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Example. $\mathbf{X}$ uniform over a code C of distance d.

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So, if $\mathbf{X}$ is uniform over $\mathcal{C}$, then

$$|\mathcal{C}| = \frac{1}{F_{\mathbf{X}}(d)}.$$

In fact, this is tight.

Theorem (Main theorem)

Let $M\ \ (d)$ be the optimal size of a distance d code. Then

$$M\ \ (d) = \sup_{\mathbf{X}} \frac{1}{F_{\mathbf{X}}(d)} = \sup_{\mathbf{X}} \frac{1}{\Pr\ \ (\mathbf{X};\hat{\mathbf{X}})}$$

Exact distance spectrum formula

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Key points:

- No asymptotics!
- Exact formula for basically any distance measure.

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- Turns question about codes into one about distributions.
- Allows us to use optimization techniques for distributions.
- New bounds on the second-order asymptotics.
- Best distribution is uniform over optimal code, but any distribution gives a lower bound.

Proof for Discrete Case

For a fixed random vector **X**, want to show:

$$F_{\mathbf{X}}(d) = \Pr[\ (\mathbf{X};\hat{\mathbf{X}}) < d] \quad \frac{1}{M\ (d)}.$$

Two steps:

1. If $|\mathrm{supp}(\mathbf{X})| = M \quad M\ (d)$, then

$$F_{\mathbf{X}}(d) \quad \frac{1}{M\ (d)}.$$

Step 1: small support

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$$\Pr\quad(\mathbf{X};\hat{\mathbf{X}}) < d \qquad \sum_{\mathbf{x}\in\mathrm{supp}(\mathbf{X})} P_{\mathbf{X}}(\mathbf{x})^2.$$

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Assume $j\mathrm{supp}(\mathbf{X})j = M \quad M \ (d)$. Then

$$\frac{1}{M} \underset{\mathbf{x}2\mathrm{supp}(\mathbf{X})}{\mathrm{X}} P_{\mathbf{X}}(\mathbf{x}) = \frac{1}{M}:$$

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By Cauchy-Schwartz,

$$\mathop{\mathrm{X}}\limits_{\mathbf{x}2\mathrm{supp}(\mathbf{X})} P_{\mathbf{X}}(\mathbf{x})^2 \qquad \mathop{\mathrm{X}}\limits_{\mathbf{x}2\mathrm{supp}(\mathbf{X})} \frac{1}{M^2} = \frac{1}{M} \qquad \frac{1}{M \ (d)} :$$

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So, for small support, uniform is best.

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Showed that if $j\mathrm{supp}(\mathbf{X})j$ is small, $F_{\mathbf{X}}(d) \qquad \frac{1}{M\ \ (d)}$.

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Idea: If $j\mathrm{supp}(\mathbf{X})j$ is large, show how to reduce $j\mathrm{supp}(\mathbf{X})j$ without increasing $F_{\mathbf{X}}(d)$.

Specifically, we'll find $\mathbf{X}^{0}$ with support size

$$j\mathrm{supp}(\mathbf{X})j \quad 1$$

and

$$F_{\mathbf{X}^{0}}(d) \quad F_{\mathbf{X}}(d):$$

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$$F_{\mathbf{X}^{0}}(d) \quad F_{\mathbf{X}}(d):$$

If we iterate this until the support has size $M\ \ (d)$, then

$$F_{\mathbf{X}}(d) \quad F_{\mathbf{X}^{0}}(d) \quad F_{\mathbf{X}^{00}}(d) \qquad \frac{1}{M\ \ (d)}:$$

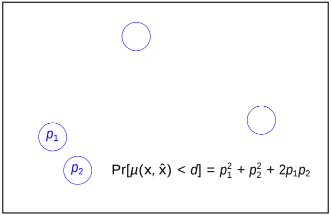
Large support cont.

Support reduction. Starting with distribution **X** on large support $M > M^X$

Large support cont.

Support reduction. Starting with distribution $\mathbf{X}$ on large support $M > M \ (d)$, want to construct $\mathbf{X}^0$ on smaller support.

Intuition $\Pr[\ (\mathbf{X}; \hat{\mathbf{X}}) < d] = \overset{\mathrm{P}}{\ }_{i;j}\, p_i p_j \mathbf{1} f \ (\mathbf{x}_i; \mathbf{x}_j) < d g$ where $p_i = P_{\mathbf{X}}(\mathbf{x}_i)$



$\Pr[\mu(x, \hat{x}) < d] = p_1^2 + p_2^2 + 2p_1p_2$

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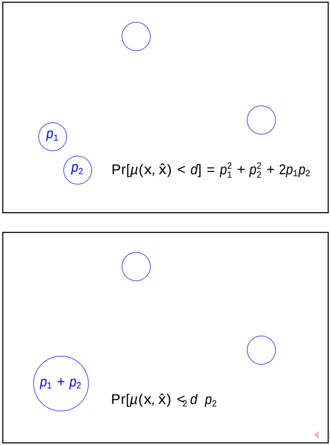
p_1
p_2
$\Pr[\mu(x, \hat{x}) < d] = p_1^2 + p_2^2 + 2p_1p_2$
$p_1 + p_2$
$\Pr[\mu(x, \hat{x}) <_2 d\ \ p_2$

Large support cont.

Support reduction. Starting with distribution $\mathbf{X}$ on large support $M > M\ (d)$, want to construct $\mathbf{X}'$ on smaller support.

Large support cont.

Support reduction. Starting with distribution **X** on large support $M > M \ (d)$, want to construct $\mathbf{X}^{0}$ on smaller support.

Proof.

If $j\text{supp}(\mathbf{X})j > M \ (d)$ (**X**.

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Proof.
If $j\mathrm{supp}(\mathbf{X})j > M\ \ (d)$, have $\mathbf{x}; \mathbf{y}\ 2\ \mathrm{supp}(\mathbf{X})$ at distance $< d$. Want to "combine" $\mathbf{x}; \mathbf{y}$.

Question: Which of $\mathbf{x}; \mathbf{y}$ to keep?

Answer: "Furthest": Keep $\mathbf{x}$ if

$$\Pr\quad(\mathbf{x}; \mathbf{X}) < d \qquad \Pr\quad(\mathbf{y}; \mathbf{X}) < d\ :$$

Large support cont.

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(Upper bound via uniform distribution.)

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“Support reduction” proof is (sort of) constructive.

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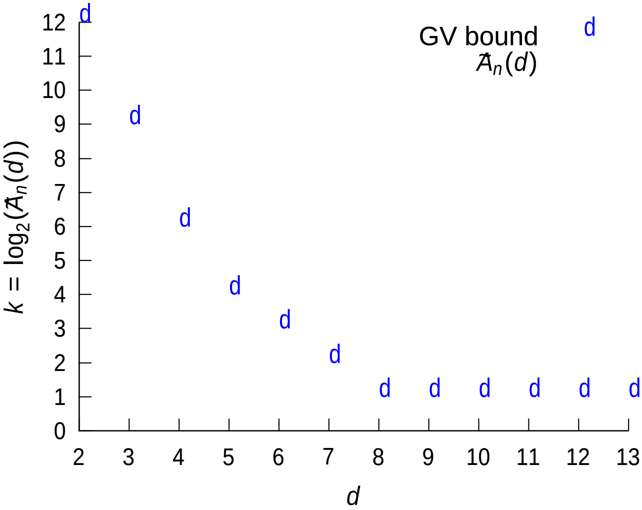
Start with any distribution, look at two codewords at distance $< d$, remove the one which is “closer” to the code.

An Algorithmic Construction ($n = 13$)



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  1. T. S. Motzkin and E. G. Straus, “Maxima for graphs and a new proof of a theorem of Turan,” Canad. J. Math, vol. 17, no. 4, pp. 533–540, 1965.
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- We now generalize to the case in which $jXj = 1$ (even uncountable)

- Idea: Greedy selection of codewords $f\mathbf{u}_i g_{i=1}^{k}$ given a fixed random vector/distribution $\mathbf{X} \quad P_{\mathbf{X}}$.

Non-Discrete Code Alphabets: Illustration

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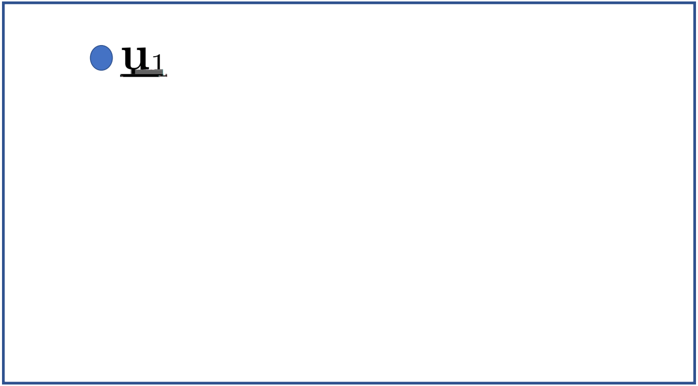

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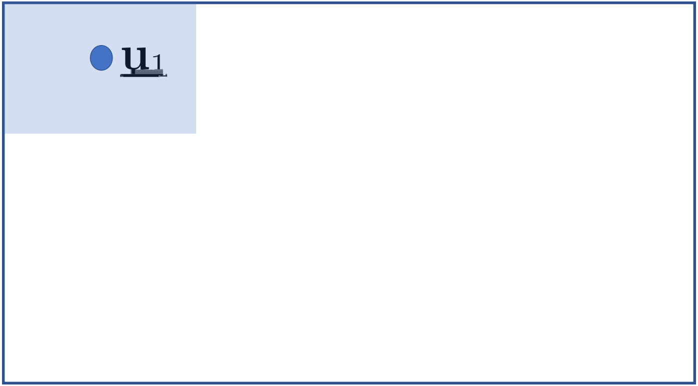

$$\mathbf{u}_1 = \arg\min_{\mathbf{u}_1} \Pr \; \mathbf{X} \in B_d(\mathbf{u}_1)$$

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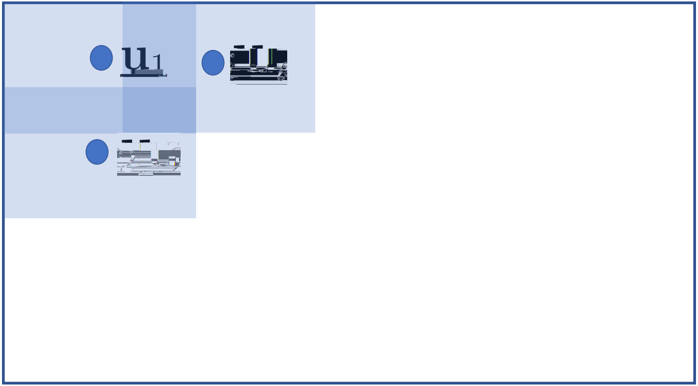

$$\mathbf{u}_i = \arg\min_{\mathbf{u}_i} \Pr\left(\mathbf{X} \in B_d(\mathbf{u}_i) \setminus \cup_{j=1}^{i-1} B_d(\mathbf{u}_j)\right)$$

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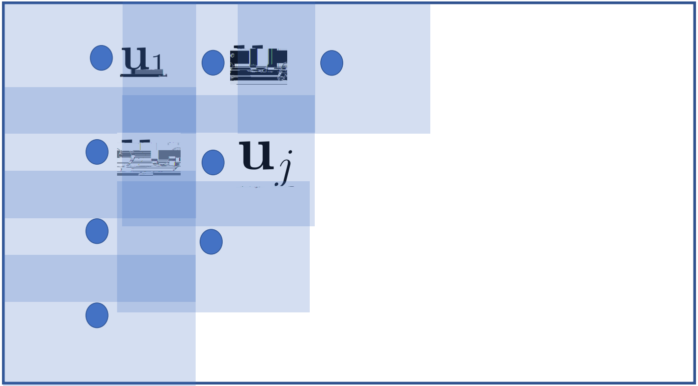

Until you run out of space!

Non-Discrete Code Alphabets: Achievability Proof

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The code $C = f\mathbf{u}_i : i = 1; : : : ; Mg$ formed is a distance-d code and

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Let $D_i := B_d(\mathbf{u}_i) \setminus \cup_{j=1}^{i-1} B_d(\mathbf{u}_j)$ and note that $\{D_i\}$ forms a partition of $\mathcal{X}^n$.

$$\begin{aligned}
\Pr[\rho(\mathbf{X}, \hat{\mathbf{X}}) < d] &= \sum_{j=1}^{M} \int_{\mathbf{x} \in D_j} \left(\int_{\hat{\mathbf{x}} \in B_d(\mathbf{x})} dP_{\mathbf{X}}(\hat{\mathbf{x}}) \right) dP_{\mathbf{X}}(\mathbf{x}) \qquad \because \ \mathbf{X} \perp\!\!\!\perp \hat{\mathbf{X}} \\
&\geq \sum_{j=1}^{M} \int_{\mathbf{x} \in D_j} p_j \, dP_{\mathbf{X}}(\mathbf{x}) \qquad \because \ \min_{\mathbf{x} \in D_j} P_{\mathbf{X}}\{B_d(\mathbf{x})\} \geq p_j
\end{aligned}$$

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$$\Pr[\;\; (\mathbf{X}; \hat{\mathbf{X}}) < d] = \overset{M}{\underset{j=1}{X}} \; \underset{\mathbf{x} 2 D_j}{Z} \;\; \underset{\hat{\mathbf{x}} 2 B_d(\mathbf{x})}{Z} \; dP_{\mathbf{X}}(\hat{\mathbf{x}}) \;\; ! \; dP_{\mathbf{X}}(\mathbf{x}) \qquad * \;\; \mathbf{X} \; ?? \; \hat{\mathbf{X}}$$

$$\overset{M}{\underset{j=1}{X}} \; \underset{\mathbf{x} 2 D_j}{Z} \; p_j \, dP_{\mathbf{X}}(\mathbf{x}) \qquad * \;\; \min_{\mathbf{x} 2 D_j} P_{\mathbf{X}} f B_d(\mathbf{x}) g \qquad p_j$$

$$\overset{M}{\underset{j=1}{X}} \; p_j^2 \qquad \frac{1}{M} \qquad \frac{1}{M \;\; (d)} \qquad * \;\; \text{Cauchy-Schwarz \& } M \qquad M \;\; (d)$$

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- But we removed space $B_d(\mathbf{u}_k) \quad X^n$ successively instead of codewords successively.
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$$F_{\mathbf{X}}(d) = \Pr \quad (\mathbf{X};\hat{\mathbf{X}}) < d \qquad \frac{1}{M\ (d)} \qquad =) \qquad M\ (d) \qquad \sup_{\mathbf{X}} \frac{1}{F_{\mathbf{X}}(d)}:$$

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- In summary,

$$M \ (d) = \sup_{\mathbf{X}} \frac{1}{F_{\mathbf{X}}(d)}$$

Refined Asymptotics I

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Refined Asymptotics II

Corollary (Upper Bound on Rate)

For any arbitrary bounded distance measure, the optimal code rate for distance n is

$$R_n(\ ) \quad I_{X^n}(\ ) + O\left(\frac{1}{\sqrt{n}}\right) \ :$$

where the large-deviations rate function is

$$I_{X^n}(a) := \sup\{a \quad \varphi_{X^n}(\ )\} ; \quad \textit{and} \quad \varphi_X(\ ) := \log \mathbb{E}\left[e^{\ (X;\hat{X})}\right] :$$

Proof.

Careful tilting of probability distributions. □

First-Order Asymptotics

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Corollary (First-Order Asymptotics on Rate)

If the sequence of distance measures satisfies

$$\sup_{n2\mathrm{N}} \max_{x^n;\hat{x}^n} \frac{1}{n} \;\;(x^n;\hat{x}^n) < \;1\,;$$

then we have

$$\limsup_{n\,!\,1} R_n(\;\;) = \limsup_{n\,!\,1} I_{X^n}(\;\;); \quad \textit{and}$$

$$\liminf_{n\,!\,1} R_n(\;\;) = \liminf_{n\,!\,1} I_{X^n}(\;\;)$$

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New derivation of Hamming bound

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Corollary (Hamming Bound for Finite jXj)

$$M \quad (d) \qquad \inf_{>0} \frac{jXj^n}{B_{(d \quad)=2}(\mathbf{0})} \qquad \frac{jXj^n}{B_{b(d \quad 1)=2c}(\mathbf{0})}$$

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Proof: (Due to V. Guruswami).

Let $e = (d \qquad)=2$. Then

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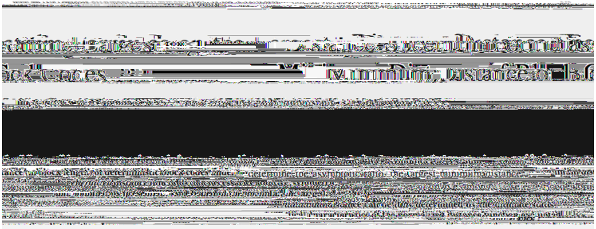
$$\underset{\mathbf{x}}{X}\ \underset{\mathbf{y}2B_e(\mathbf{x})}{X}\ \underset{\mathbf{z}2B_e(\mathbf{x})}{X} P_{\mathbf{X}}(\mathbf{y})P_{\mathbf{X}}(\mathbf{z})$$

$$\text{CS} \quad \underset{\mathbf{x}}{X}\ \underset{\mathbf{y}2B_e(\mathbf{x})}{X} P_{\mathbf{X}}(\mathbf{y})\ !^{\,2} = \frac{jB_e(\mathbf{0})j^2}{jXj^n}$$

Related Work



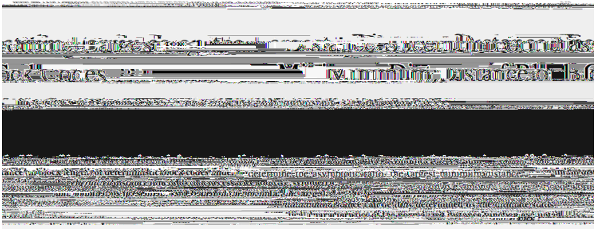
[illegible] Largest [illegible] Distance [illegible]

[illegible] Block Codes [illegible] Minimum Distance of Block [illegible]

[illegible] block length of deterministic block codes [illegible] determine the asymptotic ratio [illegible] the largest minimum distance [illegible]

[illegible] minimum distance can be [illegible] determined [illegible]

Related Work

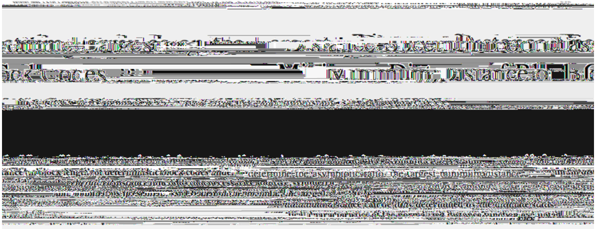


[illegible]

My visit to NCTU in 2015

- Chen, Lee and Han (2000) proved an elegant information spectrum-style result

Related Work



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Thanks!