Information Rates for Phase Noise Channels

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Outline

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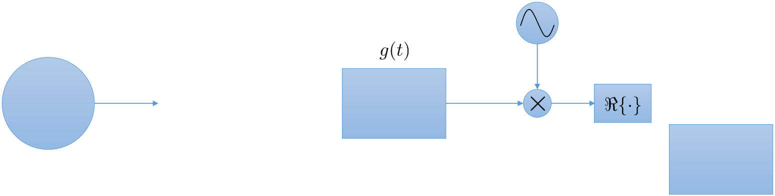
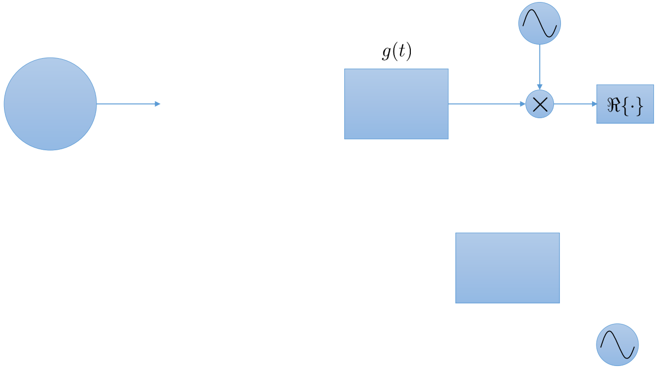
1. Motivation
2. From continuous to discrete time
3.

A Classic Communication Scheme



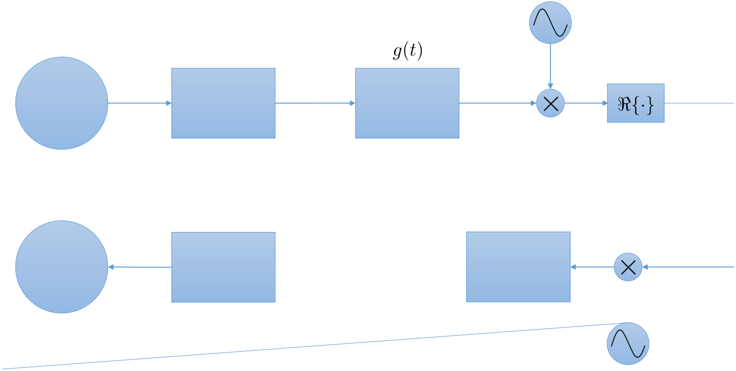
The AWGN Channel



$$Y_k = X_k + W_k$$

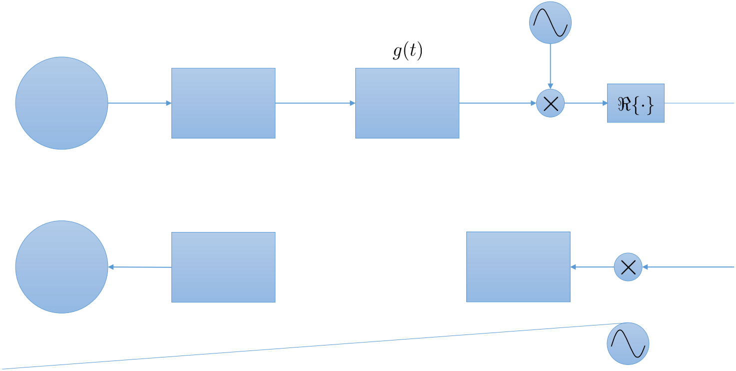
The Actual

The Actual AWGN Channel



Phase noise processes tx and rx

The Actual AWGN Channel



Phase noise processes $_{\text{tx}}$ and $_{\text{rx}}$
Sampling at kT_{symb} is no longer optimal!

Phase Noise Channels

Phase noise arises due to

- Imperfections in the oscillator circuits at the transceivers

Phase Noise Channels

Phase noise arises due to

- Imperfections in the oscillator circuits at the transceivers
  - Even for high-quality oscillators: if the continuous-time waveform is processed by long lters at the receiver (e.g., long symbol time, OFDM systems), the phase uncertainty accumulates!

Phase Noise Channels

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Phase Noise Channels

Questions:

- Models for phase noise channels

Phase Noise Channels

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-

Phase Noise Channels

Questions:

- Models for phase noise channels
- Impact of phase noise on channel capacity
- Signal and code design

Outline

Representation of continuous-time waveforms

- Assume input waveform $\{X(t)\}_{t=0}^{T}$ is square integrable:

$$X(t) \in L^2[0;T] \quad \rightarrow \quad \int_0^T |X(t)|^2 dt < \infty$$

Representation of continuous-time waveforms

- Assume input waveform $\{X(t)\}_t^T$

Mutual information for random waveforms

- The average mutual information between $\{X(t)\}_{t=0}^{T}$ and $\{Y(t)\}_{t=0}^{T}$ is [Gallager, 1968]

$$I\left(\{X(t)\}_{t=0}^{T}; \{Y(t)\}_{t=0}^{T}\right) = \lim_{n \to \infty} I(X_1 \ldots X_n; Y_1 \ldots Y_n)$$

(if it exists)

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Example: mutual information for a phase noise channel

- Input-output relation:

$$Y(t) = X(t)e^{j\quad(t)} + W(t)$$

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- Input-output relation:

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- Choose an incomplete orthonormal basis as

$$_m(t) = \frac{1}{\sqrt{\quad}}\mathrm{rect}\left(\frac{t \quad m \quad + \quad /2}{\quad}\right); \qquad m = 1,\ldots,n; \qquad = T/n$$

Example: mutual information for a phase noise channel

- Discretized model:

$$Y_m = X_m \int_{(m\ \ 1)}^{m} \frac{e^{j\ \ (t)}}{\ } dt + W_m$$

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- Both amplitude fading and phase noise!
- Commonly used discrete-time phase noise channel model:

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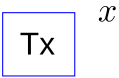
- How di erent are the two models in terms of capacity?

Receivers with nite time resolution



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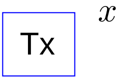

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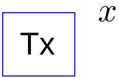

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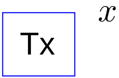

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- Oversampling helps! $\& \quad =) \quad I(X_1 \quad X_n ; Y_1 \quad Y$

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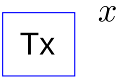

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Oversampled channel model

$$Y_{mL+\ell} = X_m F_{mL+\ell} + W_m; \quad \ell = 1;\dots;L$$
$$m = 1;\dots;n$$

$$F_{mL+\ell} = \int_{(mL+\ell-1)\Delta}^{(mL+\ell)\Delta} \frac{e^{j\theta(t)}}{\Delta} dt; \qquad \Delta = \frac{T_{\text{symb}}}{L}$$

Wiener phase noise

- Widely used for modeling oscillators
- Also known as Brownian motion:

$$(t) = \quad (0) + \int_0^t Z(t')dt'$$

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- Variance increases: $(t) \quad N(0; \; ^2 t)$
- Samples are not independent: $\mathrm{E}[\;(t)\;(s)] = \; ^2 \min\{t; s\}$
- Process with memory!
- Oversampling, i.e. $L > 1$, increases information rates

Wiener phase noise channel

De ne $_k =$ $((k$ $1)$ $)$ and N_k $N(0;1)$:

$$_k = \;_{k\;\;1} + \; p\overline{}\, N_k$$

Wiener phase noise channel

De ne $_k = \ ((k \quad 1) \quad)$ and $N_k \quad N(0;1)$:

$$ {}_k = \quad {}_{k \quad 1} + \sqrt{\ }\, N_k $$

$$ Y_{mL+\ell} = X_m e^{j \quad {}_{mL+\ell}} \int_{(mL+\ell \quad 1)}^{(mL+\ell)} \frac{e^{j(\quad (t) \quad {}_{mL+\ell})}}{\ } \mathrm{d}t + W_{mL+\ell} $$

Wiener phase noise channel

De ne $_k$ = ((k 1)) and N_k $N(0; 1)$:

$$_k = _{k \; 1} + p\text{—} N_k$$

$$Y_{mL+`} = X_m e^{j \; mL+`} \int_{(mL+` \; 1)}^{(mL+`)} \frac{e^{j(\; (t) \; mL+`)}}{} \mathrm{d}t + W_{mL+`}$$

Contour plot of the unnormalized fading pdf for = 6 and = 1. (Y. Wang *et al.*, TCOM 2006, vol. 54, no. 5)

Oversampled Discrete-time Wiener Phase Noise Channel

- Let us study the capacity of a simpler model

$$ {}_k = {}_{k\ \ 1} + p\text{—}N_k $$

$$ Y_{mL} $$

Oversampled Discrete-time Wiener Phase Noise Channel

- Let us study the capacity of a simpler model

$$\theta_k = \theta_{k-1} + \sqrt{\gamma} N_k$$

$$Y_{mL+\ell} = X_m e^{j\theta_{mL+\ell}} + W_{mL+\ell}$$

- Capacity under an average power constraint:

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T |X(t)|^2 \, dt \le P$$

Oversampled Discrete-time Wiener Phase Noise Channel

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$$\mathbb{E}\ \frac{1}{T}\int_0^T |X(t)|^2\,\mathrm{d}t \quad P \quad \Longrightarrow \quad \mathbb{E}\ |X_m|^2 \quad P$$

How to compute capacity

Challenges:

- Channel with memory

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We will see how to:

- Get rid of the memory
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We will see how to:

- Get rid of the memory
- Compute an upper bound to capacity
- We assume iid X_m's with $\underline{\angle X_m} \quad U[0;2\ )$

A capacity upper bound

Define $X_1^n = X_1 \ldots X_n$

$$\frac{1}{n} I\left(X_1^n ; \mathbf{Y}_1^n\right) = \frac{1}{n} \sum_{m=1}^{n} I\left(X_1^n ; \mathbf{Y}_m \mid \mathbf{Y}_1^{m-1}\right)$$

A capacity upper bound

De ne $X_1^n = X_1 \quad X_n$

$$\frac{1}{n} I\,(X_1^n\,;\mathbf{Y}_1^n) = \frac{1}{n}\sum_{m=1}^{n} I \quad X_1^n;\mathbf{Y}_m\, j\,\mathbf{Y}_1^{m \quad 1}$$

$$\text{(DPI)} \qquad \frac{1}{n}\sum_{m=1}^{n} I \quad X_1^n; \quad {}_{mL+1};\mathbf{Y}_m\, j\,\mathbf{Y}_1^{m \quad 1}$$

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$$= \frac{1}{n} \sum_{m=1}^{n} I\left(X_1^n; \mathbf{Y}_m \,j\, {}_{mL+1}; \mathbf{Y}_1^{m-1}\right) + I\left({}_{mL+1}; \mathbf{Y}_m \,j\, \mathbf{Y}_1^{m-1}\right)$$

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$$\text{(Markov)} = \frac{1}{n}\sum_{m=1}^{n} I\left(X_m; \mathbf{Y}_m \,|\, \quad {}_{mL+1}\right) + I \quad {}_{mL+1}; \mathbf{Y}_m \,|\, \mathbf{Y}_1^{m \quad 1}$$

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$$\text{(Stationary)} = I(X_1;\mathbf{Y}_1 \,|\, \quad _{L+1}) + \frac{1}{n}\sum_{m=1}^{n} I \quad\quad _{mL+1};\mathbf{Y}_m \,|\, \mathbf{Y}_1^{m \quad 1}$$

A capacity upper bound

Assumption: iid X_m's with $\underline{\angle X_m} \quad U[0; 2\ )$

$$I \quad {}_{mL+1}; \mathbf{Y}_m j \, \mathbf{Y}_1^{m \quad 1}$$

$$\text{(DPI)} \quad I \quad {}_{mL+1}; \quad {}_{mL+1} \quad \underline{\angle X_m} \quad p\,\overline{}\, N_{mL+1} \quad \mathbf{Y}_1^{m \quad 1}$$

$$= 0$$

Polar decomposition

$$
\begin{aligned}
I(X_1; \mathbf{Y}_1 | \;_{L+1}) &= I(|X_1|; \angle X_1; \mathbf{Y}_1 | \;_{L+1}) \\
&= \underbrace{I(|X_1|; \mathbf{Y}_1 | \;_{L+1})}_{\text{amplitude mod.}} + \underbrace{I(\angle X_1; \mathbf{Y}_1 | \;_{L+1}; |X_1|)}_{\text{phase mod.}}
\end{aligned}
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Polar decomposition

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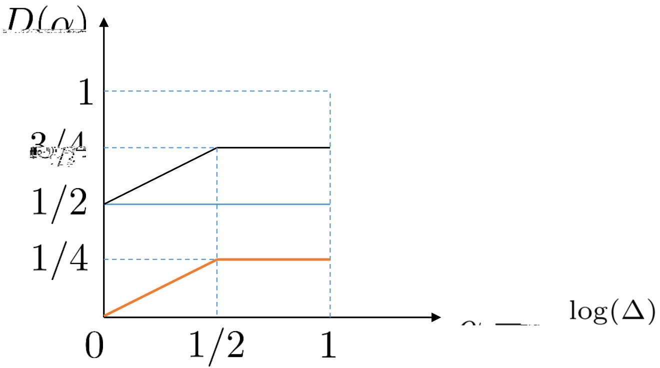
- Different bounding techniques are used for the two terms
  - Reveal all phase noise samples to the receiver: amplitude mod. on AWGN channel
  - Application of the I-MMSE formula to the phase mod. term

$$C \quad \frac{1}{2}\log\left(1+\frac{P}{2}\right) + \log(2\;) + \frac{1}{2}\log\left(P\left(\sqrt{1+\frac{4}{{}^2P\;{}^2}}\quad 1\right)\right)$$

Degrees of freedom

Define the degrees of freedom as

$$D(\;) = \lim_{P \to \infty} \frac{C(P;\quad = P\quad;\;)}{\log(P)}$$



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    - For any > 0! Even for very high-quality oscillators!
- Conjectures:
    - Simplifying the model by discarding the amplitude fading is too much
    - The fundamental tension between additive noise and phase noise limits the degrees of freedom

Thanks for your attention!

Derivation of the average power constraint

$$\lim_{T \to \infty} \frac{1}{T}$$